Worcester County Mathematics League

Varsity Meet 1
October 8, 2014

COACHES' COPY
ROUNDS, ANSWERS, AND SOLUTIONS

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 1 – October 8, 2014
Round 1: Arithmetic

All answers must be in simplest exact form in the answer section
NO CALCULATOR ALLOWED

1. Simplify:

$$\frac{(2\times10^{5})^{2}}{(8\times10^{-3})(5\times10^{12})}$$

2. If $a \star b = a^2 b$ and $a \triangle b = a + \frac{1}{b}$, evaluate $(a \star b) \triangle (b \star a)$ when $a = 2$ and $b = \frac{1}{3}$.

3. A certain binary operator, denoted ♥, satisfies the following three conditions for all real numbers x and y:

i. $x \heartsuit 0 = x$
ii. $x \heartsuit y = y \heartsuit x$
iii. $(x+1) \heartsuit y = (x \heartsuit y) + y + 1$

Find the value of $12 \heartsuit 4$.

ANSWERS

(1 pt.) 1. ______________________

(2 pts.) 2. ______________________

(3 pts.) 3. ______________________

Varsity Meet 1 – October 8, 2014
Round 2: Algebra 1

All answers must be in simplest exact form in the answer section
NO CALCULATOR ALLOWED

1. Solve for x:

$$\frac{1}{2}(x-5)-\frac{x}{3}+10=\frac{1}{5}(3x+2)$$

2. A two-digit counting number has a value that is 13 greater than 3 times the sum of its digits. If the units digit is one greater than the tens digit, what is the number?

3. Working alone, Barb takes three times as long as Emily to complete a certain job. If they share the job, it can be completed when Emily works for $\frac{2}{3}$ of a day and Barb works for 7 days. How long does it take Emily to do the job alone?

ANSWERS

(1 pt.) 1. $x =$ ____________________

(2 pts.) 2. ________________________

(3 pts.) 3. ___________________ days

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Round 3: Set Theory

All answers must be in simplest exact form in the answer section
NO CALCULATOR ALLOWED

1. Let $A = \{3, 6, 8, 10\}$, $B = \{3, 5, 7, 11\}$, and $C = \{6, 7, 8, 11\}$. Find:

$$[(A \cap B) \cup (B \cap C)] \cup (A \cap C)$$

2. At a particular school, 25 students play baseball, 40 play football, and 20 play hockey. Of these students, nine play both hockey and baseball, thirteen play both hockey and football, and eleven play both baseball and football. Of those students who play two sports, there are 5 who play all three. If there are 100 students at the school, how many do not play any sport?

3. Let X be the set containing all subsets of $\{M, A, T, H\}$ and let Y be the set containing all subsets of $\{T, E, A, M\}$. If p is the number of elements in $X \cap Y$, and q is the number of elements in $(X \cap \bar{Y}) \cup (Y \cap \bar{X})$, where the universal set is $X \cup Y$, find the value of $\frac{p}{q}$.

ANSWERS

(1 pt.) 1. { ____________________ }

(2 pts.) 2. ________________ students

(3 pts.) 3. ______________________

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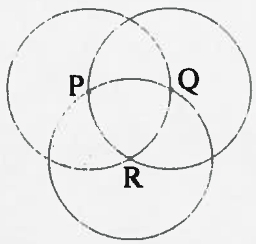
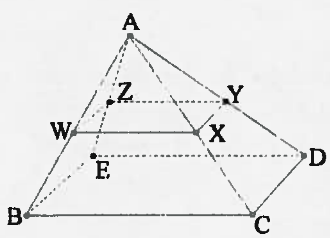
Round 4: Measurement

All answers must be in simplest exact form in the answer section
NO CALCULATOR ALLOWED

1. 

P, Q, and R are the centers of the three circles in the figure. If the segment PQ is 2 units, find the area of the shaded region.

2. A square has a side of length m and a regular hexagon has a side of length n. The hexagon's area is 4 times the area of the square. Find $\frac{m^4}{n^4}$.

3. 

The solid figure $ABCDE$ is a right square pyramid. The points W, X, Y, and Z are the respective midpoints of $\overline{AB}$, $\overline{AC}$, $\overline{AD}$, and $\overline{AE}$. If $AY = 2$ units and $BC = AC$, find the volume of the solid figure (called a *frustum*) $BCDEZYXW$.

ANSWERS

(1 pt.) 1. _______________ square units

(2 pts.) 2. _________________________

(3 pts.) 3. ________________ cubic units

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Round 5: Polynomial Equations

All answers must be in simplest exact form in the answer section
NO CALCULATOR ALLOWED

1. Factor $3x^5 - 15x^3 + 12x$ completely.

2. Find the sum of the reciprocals of the roots of the equation $x^2 + ax + b = 0$.

3. The polynomial $f(x) = x^4 - 4x^2 + x - 6$ has four distinct roots, each of which is half the value of one of the roots of $g(x) = x^4 + ax^3 + bx^2 + cx + d$.
Find a, b, c, and d.

ANSWERS

(1 pt.) 1. ______________________

(2 pts.) 2. ______________________

(3 pts.) 3. $a =$_______, $b =$_______, $c =$_______, $d =$_______

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Team Round

All answers must be in simplest exact form, and be written on the separate team answer sheet

1. Find the area of the polygon bounded by the solution set to $|x| + |4y| = 8$.

2. If the diagonal of a rectangle is $4\sqrt{3}$ and its area is 8, what is its perimeter?

3. ΔABC has vertices at the points $(4, 4)$, $(4, 8)$, and $(12, 4)$. ΔXYZ has vertices at the points $(6, 5)$, $(6, 13)$, and $(10, 5)$. Determine the value of the ratio:

$$\frac{Area(\Delta ABC \cap \Delta XYZ)}{Area(\Delta ABC \cup \Delta XYZ)}$$

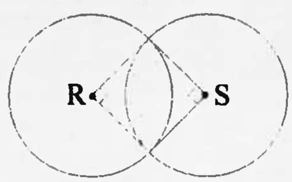
4. Let $S_1 = \{(x, y) : y = |x|\}$, $S_2 = \{(x, y) : y = 3x^5 - 8x^4 - x^3 + 6x^2 - 2x + 3\}$, and $S_3 = \{(x, y) : y = x^2\}$. Find $(S_1 \cap S_2) \cap S_3$.

5. Find the largest value of p for which the points $(11, 2)$, $(9, p)$, and $(-13, p^2 - p)$ will be collinear.

6. Determine the value of:

$$4 - \cfrac{4}{4 - \cfrac{4}{4 - \cfrac{4}{4 - \ddots}}}$$

7. Find the value of k so that when $2x^4 + 5x^3 + kx^2 + 11x + 10$ is divided by $x + 3$ the remainder is 2.

8. 

The circles R and S each have a radius of 5 units. The area of the shaded region can be expressed as $a\pi + b$ square units where a and b are rational numbers. Find $a + b$.

9. How many distinct pairs (x, y) of real numbers satisfy both $x = x^2 + y^2$ and $y = 2xy$?

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Team Round Answer Sheet

ALL ANSWERS MUST BE IN SIMPLEST EXACT FORM, AND BE WRITTEN ON THIS TEAM ANSWER SHEET.

(2 points each)

1. ________________ square units

2. ____________________________

3. ____________________________

4. ____________________________

5. ____________________________

6. ____________________________

7. ____________________________

8. ____________________________

9. ____________________________

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Answers

Round 1: Arithmetic
Tantasqua, St. John's, Algonquin

1. 1
2. $\frac{35}{6}$ or $5\frac{5}{6}$
3. 64

Round 2: Algebra 1
Quaboag, Tahanto, Bancroft

1. $\frac{213}{13}$ or $16\frac{5}{13}$
2. 34
3. 3

Round 3: Set Theory
West Boylston, Clinton, Quaboag

1. $\{3, 6, 7, 8, 11\}$
2. 43
3. $\frac{1}{2}$ or 0.5

Round 4: Measurement
Burncoat, Auburn, Westborough

1. $\frac{2}{3}\pi$
2. $\frac{27}{64}$
3. $\frac{28}{3}\sqrt{2}$ or $9\frac{1}{3}\sqrt{2}$

Round 5: Polynomial Equations
Leicester, St. John's, Auburn

1. $3(x-2)(x-1)x(x+1)(x+2)$
2. $-\frac{a}{b}$
3. $a = 0, b = -16, c = 8, d = -96$

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ANSWERS: Team Round

Quaboag, Westborough, Doherty, Nashoba, Shrewsbury, Worcester Academy, Algonquin, Bromfield, St. John's

1. 32

2. 16

3. $\frac{1}{7}$

4. $\{(-1,1),(1,1)\}$

5. 11

6. 2

7. $-\frac{2}{9}$

8. $-\frac{25}{2}$ or $-12\frac{1}{2}$ or -12.5

9. 4

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Solutions

Round 1: Arithmetic

1. $\frac{(2\times10^5)^2}{(8\times10^{-3})(5\times10^{12})} = \frac{4\times10^{10}}{40\times10^9} = \frac{1}{10}\times10^1 = \boxed{1}$

2. $\left(2 \star \frac{1}{3}\right) \Delta \left(\frac{1}{3} \star 2\right) = \left(2^2\times\frac{1}{3}\right) \Delta \left[\left(\frac{1}{3}\right)^2 \times 2\right] = \frac{4}{3} \Delta \frac{2}{9} = \frac{4}{3} + \frac{9}{2} = \boxed{\frac{35}{6}}$

3. By (ii), $12 \heartsuit 4 = 4 \heartsuit 12$.
By (iii), $4 \heartsuit 12 = (3 \heartsuit 12) + 13 = (2 \heartsuit 12) + 2 \times 13 = (1 \heartsuit 12) + 3 \times 13$
$= (0 \heartsuit 12) + 4 \times 13$.
By (i), this equals $12 + 4 \times 13 = \boxed{64}$

Round 2: Algebra 1

1. $\frac{1}{2}(x-5) - \frac{x}{3} + 10 = \frac{1}{5}(3x+2) \Rightarrow \frac{1}{2}x - \frac{5}{2} - \frac{1}{3}x + 10 = \frac{3}{5}x + \frac{2}{5} \Rightarrow \frac{1}{6}x + \frac{15}{2} = \frac{3}{5}x + \frac{2}{5} \Rightarrow \frac{71}{10} = \frac{13}{30}x \Rightarrow x = \boxed{\frac{213}{13}}$

2. <u>Method 1</u>: If the tens digit is x and the ones digit is y, then we have $10x + y = 13 + 3(x+y)$ and $y = x + 1$. Substitution gives $10x + (x+1) = 13 + 3\big(x + (x+1)\big) \Rightarrow 11x + 1 = 6x + 16 \Rightarrow 5x = 15 \Rightarrow x = 3$. Then $y = x + 1 = 4$, and the number is $\boxed{34}$

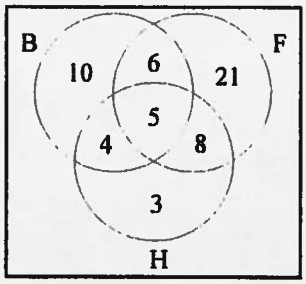
<u>Method 2</u>: By brute force, we check all two-digit numbers with a tens digit one less than the ones digit. $12 \neq 13 + 3(1+2)$, $23 \neq 13 + 3(2+3)$, but $34 = 13 + 3(3+4)$.

3. Let b be the fraction of the job that Barb can complete in one day, and let e be the fraction of the job that Emily can complete in one day. Then we have $e = 3b$ and $\frac{2}{3}e + 7b = 1$. By substitution, $\frac{2}{3}(3b) + 7b = 1 \Rightarrow 9b = 1 \Rightarrow b = \frac{1}{9}$. Then $e = 3b = \frac{1}{3}$, meaning it will take Emily $\frac{1}{\frac{1}{3}} = \boxed{3}$ days.

Round 3: Set Theory

1. $[(A \cap B) \cup (B \cap C)] \cup (A \cap C) = (\{3\} \cup \{7, 11\}) \cup \{6, 8\} = \boxed{\{3, 6, 7, 8, 11\}}$

2. In the figure, B, H, and F represent baseball, hockey, and football respectively. The sum of the numbers in the figure is 57. Then there are $100 - 57 = \boxed{43}$ students who do not play any sport.



3. Method 1: The elements in $X \cap Y$ are all the subsets of $\{M, A, T, H\} \cap \{T, E, A, M\} = \{M, A, T\}$. This set has 3 elements, so there are $2^3 = 8$ of these subsets, i.e. $p = 8$.

Since $\{M, A, T, H\}$ and $\{T, E, A, M\}$ each have four elements, both X and Y each have $2^4 = 16$ elements. The set $(X \cap \bar{Y}) \cup (Y \cap \bar{X})$ is equal to the set $(X \cup Y) \cap \overline{(X \cap Y)}$. (In fact, both represent the *symmetric difference* of X and Y.) Since $|X| + |Y| - |X \cap Y| = |X \cup Y|$ (where $|X|$ denotes the cardinality of X), we see that $\left|(X \cup Y) \cap \overline{(X \cap Y)}\right| = |X \cup Y| - |X \cap Y| = (|X| + |Y| - |X \cap Y|) - |X \cap Y| = (16 + 16 - 8) - 8 = 16$. That is, $q = 16$.

So $\frac{p}{q} = \frac{8}{16} = \boxed{\frac{1}{2}}$

Method 2: By brute force, the sixteen elements of both X and Y could be listed and compared to find $X \cap Y$ and $(X \cap \bar{Y}) \cup (Y \cap \bar{X})$. However, this method may take longer than the allotted time.

Round 4: Measurement

1. As shown in the figure, the shaded region has an area equal to a sector of one of the three circles with a 60° central angle. Therefore its area equals $\pi \times 2^2 \times \frac{60}{360} = \pi \times 4 \times \frac{1}{6} = \boxed{\frac{2}{3}\pi}$

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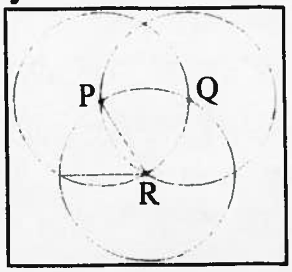
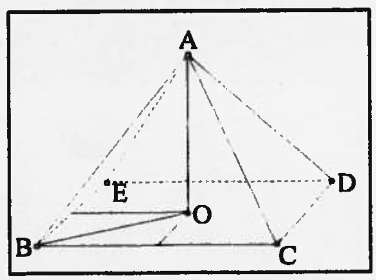


2. The area of the square is m^2 and the area of the hexagon is $\frac{3\sqrt{3}}{2}n^2$. Since the hexagon's area is four times the area of the square, we have $\frac{3\sqrt{3}}{2}n^2 = 4m^2$. Then $\frac{m^2}{n^2} = \frac{3\sqrt{3}}{8}$, and $\frac{m^4}{n^4} = \boxed{\frac{27}{64}}$

3. Since $ABCDE$ is a right square pyramid, its four triangular faces are congruent. In particular, $AB = AC = AD = AE$. Since $AY = 2$, and Y is the midpoint of $\overline{AD}$, $AB = AC = AD = AE = 4$. $BC = AC$, and the pyramid's base is a square, so $BC = CD = DE = EB = 4$.

Thus, the area of the base of pyramid $ABCDE$ is $4 \times 4 = 16$. To find its height, we apply Pythagoras' Theorem twice. First we find that the distance from a corner of the pyramid's base to the center of the base is $2\sqrt{2}$. (This is equal to the length of $\overline{OB}$, in the figure.) Second, we use this to find that the height of the pyramid is $\sqrt{4^2 - (2\sqrt{2})^2} = 2\sqrt{2}$. The volume of the pyramid is therefore $\frac{1}{3} \times 16 \times 2\sqrt{2} = \frac{32}{3}\sqrt{2}$.



The volume of the frustum is clearly equal to the volume of pyramid $ABCDE$ minus the volume of pyramid $AWXYZ$. To find the volume of the smaller pyramid, note that it is similar to the larger pyramid, and that the scale factor is $\frac{1}{2}$. (This follows from the fact that W, X, Y, and Z are the respective midpoints of $\overline{AB}$, $\overline{AC}$, $\overline{AD}$, and $\overline{AE}$.) The volume of the smaller pyramid is then equal to $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$ of the larger pyramid. Then the volume of the frustum is $\frac{32}{3}\sqrt{2} - \frac{1}{8} \times \frac{32}{3}\sqrt{2} = \boxed{\frac{28}{3}\sqrt{2}}$

Round 5: Polynomial Equations

1. $3x^5 - 15x^3 + 12x = 3x(x^4 - 5x^2 + 4) = 3x(x^2 - 1)(x^2 - 4)$
$= \boxed{3(x-2)(x-1)x(x+1)(x+2)}$

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2. Method 1: If r_1 and r_2 are the two roots of the equation, then $\frac{1}{r_1}+\frac{1}{r_2}=\frac{r_1+r_2}{r_1 r_2}$. The sum of the roots must be $-a$, and the product must be b, so $\frac{r_1+r_2}{r_1 r_2}=\boxed{-\frac{a}{b}}$

Method 2: If $-r_1$ and $-r_2$ are the two roots, then $x^2+ax+b=(x+r_1)(x+r_2)=x^2+(r_1+r_2)x+r_1r_2$, so that $a=r_1+r_2$ and $b=r_1r_2$. Then the sum of the roots' reciprocals is $\frac{1}{-r_1}+\frac{1}{-r_2}=\frac{r_1+r_2}{-r_1r_2}=\boxed{-\frac{a}{b}}$

3. Method 1: Let r_1, r_2, r_3, and r_4 be the four roots of $f(x)$. Then the four roots of $g(x)$ are $s_1=2r_1$, $s_2=2r_2$, $s_3=2r_3$, and $s_4=2r_4$. By observing the coefficients of $f(x)$, we see that $-r_1-r_2-r_3-r_4=0$, $r_1r_2+r_1r_3+r_1r_4+r_2r_3+r_2r_4+r_3r_4=-4$, $-r_1r_2r_3-r_1r_2r_4-r_1r_3r_4-r_2r_3r_4=1$, and $r_1r_2r_3r_4=-6$. By the same reasoning, $-s_1-s_2-s_3-s_4=a$, $s_1s_2+s_1s_3+s_1s_4+s_2s_3+s_2s_4+s_3s_4=b$, $-s_1s_2s_3-s_1s_2s_4-s_1s_3s_4-s_2s_3s_4=c$, and $s_1s_2s_3s_4=d$. For a, we see that $a=-s_1-s_2-s_3-s_4=-2r_1-2r_2-2r_3-2r_4=2(-r_1-r_2-r_3-r_4)$ $=2\times 0=\boxed{0.}$ Proceeding in a similar fashion, we see that $b=2^2\times -4=\boxed{-16,}$ $c=2^3\times 1=\boxed{8,}$ and $d=2^4\times -6=\boxed{-96.}$

Method 2: If r_1, r_2, r_3, and r_4 are the four roots of $f(x)$, then $f(x)=(x-r_1)(x-r_2)(x-r_3)(x-r_4)=x^4-4x^2+x-6$. If s_1, s_2, s_3, and s_4 are the roots of $g(x)$, then $g(x)=(x-s_1)(x-s_2)(x-s_3)(x-s_4)$ $=(x-2r_1)(x-2r_2)(x-2r_3)(x-2r_4)=x^4+2*0x^3-4*4x^2+8*x-16*6$, i.e. $x^4-16x^2+8x-96$. Then $\boxed{a=0}$, $\boxed{b=-16}$, $\boxed{c=8}$, $\boxed{d=-96}$.

Team Round

1. For positive values of x and y, the equation is equivalent to $x+4y=8$. The portion of this line that lies in the first quadrant (i.e. where x and y are both positive) is a segment with endpoints at $(0,2)$ and $(8,0)$.

To find the remaining solutions to the original equation, note that (x,y) is a solution if and only if each of $(-x,y)$, $(x,-y)$, and $(-x,-y)$ are solutions. Therefore the segment in the first quadrant can be flipped over the x- and y-axes to show that the graph of the full solution set is a rhombus with vertices at $(0,2)$, $(8,0)$, $(0,-2)$, and $(-8,0)$. The area of a rhombus is equal to half the product of its diagonals, so the area is $\frac{1}{2}\times 16\times 4=\boxed{32}$

Alternatively, this area can be calculated as the sum of the areas of four triangles, each having base 8 and height 2. Then the area is $4\times\left(\frac{1}{2}\times 8\times 2\right)=\boxed{32}$

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2. Let x be the height of the rectangle and y be its width. Then $x^2 + y^2 = (4\sqrt{3})^2 = 48$. The area of the rectangle is $xy = 8$. Combining these equations, we see that $x^2 + y^2 + 2xy = 48 + 2 \times 8 = 64$. Factoring this quadratic, we have $x^2 + 2xy + y^2 = (x + y)^2 = 64$, so that $x + y = \sqrt{64} = 8$. Then the perimeter equals $2(x + y) = \boxed{16}$

3. Both triangles have a width of 4 and a height of 8, and therefore an area of 16. Their intersection is the triangle with vertices at $(6,7)$, $(6,5)$, and $(10,5)$, which has width 4, height 2, and consequently area 4. Using the general formula $Area(R) + Area(S) - Area(R \cap S) = Area(R \cup S)$, we see that the area of the two triangles' union is $16 + 16 - 4 = 28$. Then the desired ratio of areas is $\frac{4}{28} = \boxed{\frac{1}{7}}$

4. The intersection operator is both commutative and associative, so $(S_1 \cap S_2) \cap S_3 = (S_1 \cap S_3) \cap S_2$. To be in $S_1 \cap S_3$, (x, y) must satisfy both $y = |x|$ and $y = x^2$. Combining these equations yields $|x| = x^2$, which is equivalent to $|x| = |x|^2$. Then $|x| = 0$ or 1, indicating that $x = -1, 0, 1$. The respective corresponding y values are 1, 0, and 1, meaning that $S_1 \cap S_3 = \{(-1,1), (0,0), (1,1)\}$. To see which of these three points are also in S_2, we substitute each ordered pair into the equation $y = 3x^5 - 8x^4 - x^3 + 6x^2 - 2x + 3$, and find that $(-1,1)$ and $(1,1)$ are solutions, while $(0,0)$ is not. Thus $(S_1 \cap S_2) \cap S_3 = \boxed{\{(-1,1), (1,1)\}}$

5. For the three points to be on the same line, the segment connecting $(11,2)$ and $(9,p)$ must have the same slope as the segment connecting $(11,2)$ and $(-13, p^2 - p)$. Thus $\frac{2-p}{11-9} = \frac{p^2-p-2}{-13-11} \Rightarrow -12(2-p) = p^2 - p - 2 \Rightarrow p^2 - 13p + 22 = 0$. This factors as $(p - 11)(p - 2) = 0$, giving $p = 11$ and $p = 2$. The greater of these two values is $\boxed{11}$

6. Let $x = 4 - \cfrac{4}{4 - \cfrac{4}{4 - \cfrac{4}{4 - \ddots}}}$. Then we note that $x = 4 - \frac{4}{x}$. Multiplying both sides by x, we have $x^2 = 4x - 4 \Rightarrow x^2 - 4x + 4 = 0 \Rightarrow (x - 2)^2 = 0$. Therefore $x = \boxed{2}$

7. Method 1: We divide as follows:

$$
\begin{array}{r l}
 & 2x^3 - x^2 + (k+3)x + (2-3k) \\
\hline
x+3 \,\big)\! & 2x^4 + 5x^3 + kx^2 + 11x + 10 \\
 & 2x^4 + 6x^3 \\
 & \quad -x^3 + kx^2 \\
 & \quad -x^3 - 3x^2 \\
 & \qquad (k+3)x^2 + 11x \\
 & \qquad (k+3)x^2 + (3k+9)x \\
 & \qquad\qquad (2-3k)x + 10 \\
 & \qquad\qquad (2-3k)x + (6-9k) \\
 & \qquad\qquad\qquad 4 + 9k
\end{array}
$$

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Setting the remainder $4 + 9k$ equal to the desired remainder 2, we solve to find that $k = \boxed{-\frac{2}{9}}$

Method 2: Using synthetic division, we have:

-3	2	5	k	11	10
		-6	3	$-3k-9$	$9k-6$
	2	-1	$k+3$	$2-3k$	$9k+4$

Then $9k + 4 = 2 \Rightarrow k = \boxed{-\frac{2}{9}}$

8. If the shaded region is split in half with a vertical segment, we see that each half of the region is equal to the portion of a 90–degree sector of one of the circles that lies outside a right triangle with both legs measuring 5 units. Therefore each half of the shaded region has an area equal to $\pi \times 5^2 \times \frac{90}{360} - \frac{1}{2} \times 5^2 = \frac{25}{4}\pi - \frac{25}{2}$. Then the full area of the region is double this, i.e. $\frac{25}{2}\pi - 25$. So $a = \frac{25}{2}$, $b = -25$, and $a + b = \boxed{-\frac{25}{2}}$

9. Starting with the equation $y = 2xy$, there are two possibilities: $y = 0$, or if not, we can divide both sides by y, giving $1 = 2x \Rightarrow x = \frac{1}{2}$. In the first case, substituting 0 for y in the first equation yields $x = x^2$, which has two solutions: $x = 0$ and $x = 1$. In the second case, substituting $x = \frac{1}{2}$ into the first equation yields $\frac{1}{2} = \frac{1}{4} + y^2$, i.e. $y^2 = \frac{1}{4}$. This has two solutions: $y = \frac{1}{2}$ and $y = -\frac{1}{2}$. So altogether, there are $\boxed{4}$ solutions.